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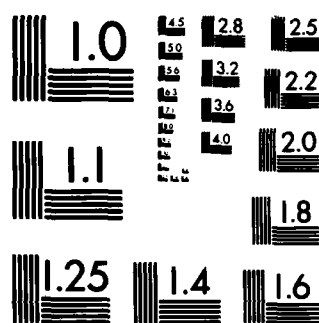
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PLANARITY CRITERIA
IN ELECTROMAGNETIC TOPOLOGY

R.S. Noss

LuTech, Inc
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Hayward CA 94545

July 1984

Final Report

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This technical report has been reviewed and is approved for publication.

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The principal tool of the electromagnetic topologist is the interaction sequence diagram (ISD), which is the dual graph of the electromagnetic topology (EMT) of a system. One of the problems of working with the ISD is its complex appearance, in part due to multiple crossings of edges. This report presents some necessary and sufficient conditions for a graph to be planar, plus an algorithm to determine the planarity of any graph from its incidence matrix. Several topological invariants of the ISD are defined to aid the discussion of computational feasibility of the algorithm.

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FIGURES

I. INTRODUCTION

The principal tool of the electromagnetic topologist is the interaction sequence diagram (ISD), which is the dual graph of the electromagnetic topology (EMT) of a system. The EMT is defined in terms of layers, sublayers, and elementary volumes (Refs. 1,2,3). Sublayers are disjointed from one another, and layers, defined to be disjointed unions of certain sublayers, are also mutually disjointed. An elementary volume shares some part of its surface with another elementary volume. All such volumes contained in a sublayer possess this property, and their union is the sublayer. If only layers and sublayers are considered, the ISD is a tree graph. In this way the complication of cycles is deferred to the elementary volume level. In the following, reference to the ISD means any subgraph corresponding to the partitioning of a sublayer into elementary volumes.

One of the problems of working with the ISD is its complex appearance, due in part to multiple crossings of edges. In some cases it is possible to reduce the number of crossings by drawing the graph differently. The ISD appearance is least complicated if its edges are drawn as straight line segments with no crossings. An intermediate step is to eliminiate the crossings. A graph which can be drawn in this way is said to be a planar graph, and such a rendition is known as a plane graph. Edges of a simple planar graph can always be represented by straight line segments which meet only at vertices.

This report presents some necessary and sufficient conditions for a graph to be planar, plus an algorithm to determine the planarity of any graph from its incidence matrix. Some definitions are introduced beforehand to facilitate the presentation of the criteria. Several topological invariants of the ISD are defined to aid the discussion of computational feasibility of the algorithm.

II. DEFINITIONS

This section presents some basic definitions, to raise the apparent ratio of theorems to definitions in the results which follow. Other definitions will be introduced as the need arises. Figures illustrating the terminology are indicated in parentheses.

A *graph* G consists of a *vertex set* V(G) of *vertices* and an *edge set* E(G) of *edges*, represented by unordered pairs of elements of V(G), called *end points* (Fig. 1). An edge becomes a *directed edge* (or *arc*) by specifying an ordered pair of vertices, called the *initial* and *terminal vertices* (Fig. 2a). If every edge in E(G) is an arc then G is a *directed graph*, or *digraph* (Fig. 2b). If functions are assigned to the edges of a graph, then a direction is implied. The opposite direction is indicated by attaching a minus sign. An example of this is current in an electrical network. In this way end points of an edge may be called initial or terminal vertices arbitrarily, and the distinction between a graph and digraph need not be stressed.

Two vertices are *adjacent* if they are end points of some edge. A vertex and edge are *incident* if the vertex is one of the end points of the edge. A *loop* is an edge or an arc that is incident with only one vertex (Fig. 3). Edges having the same end points are said to be *parallel*. A *simple graph* has no parallel edges and no loops (Fig. 4). Parallel edges are also called *multiple edges*.

Two graphs are *isomorphic* if their vertices and edges can be placed in incidence-preserving one-to-one correspondence (Fig. 5). A *geometric graph* is a graph whose vertices are selected points in two-or-three-dimensional space and whose edges are nonintersecting simple curves each of which joins two vertices (or, in the case of a loop, closes on a single vertex) without containing any other vertices. A *geometric realization* of graph G is a geometric graph that is isomorphic to G.
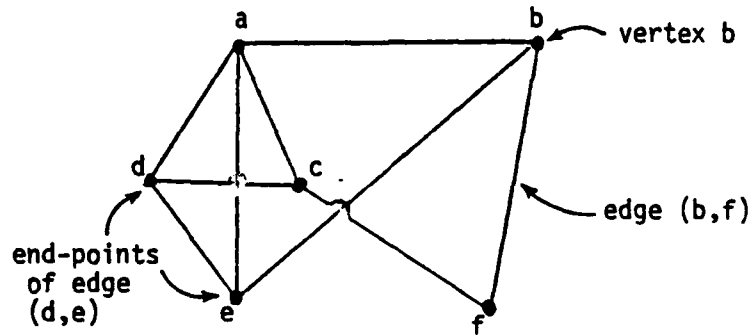
vertex set:

$$V(G) = \left\{a,b,c,d,e,f\right\}$$

edge set:

$$E(G) = \left\{(a,b),\ (a,c),\ (a,d),\ (a,e),\ (b,e),\ (b,f),\ (c,d),(c,f),\ (d,e)\right\}$$

Figure 1.   Vertex set and edge set of a graph.

terminal vertex

initial vertex
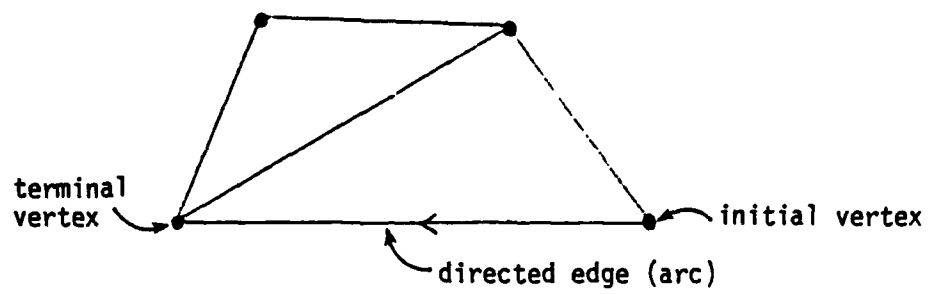
directed edge (arc)

(a) A graph containing an arc.
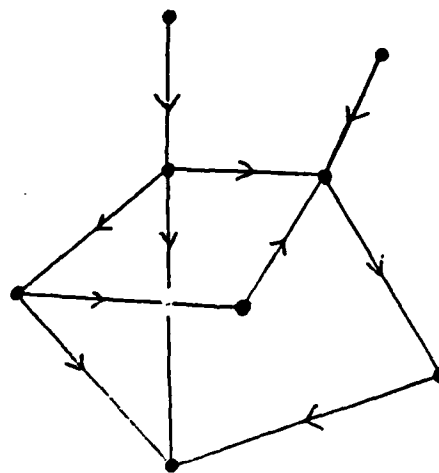
(b) A directed graph (digraph).

Figure 2. Directed edges and graphs.

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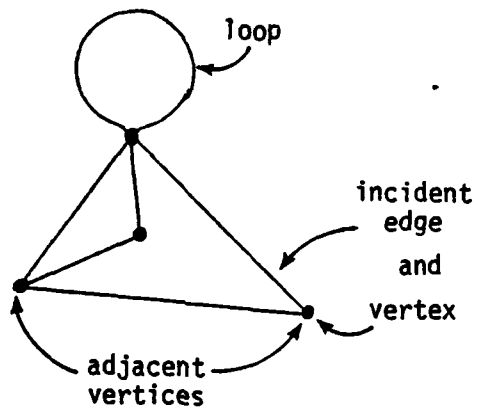
Figure 3. Adjacency, incidence, and loops.

parallel
loops

parallel
arcs

strictly parallel
arcs
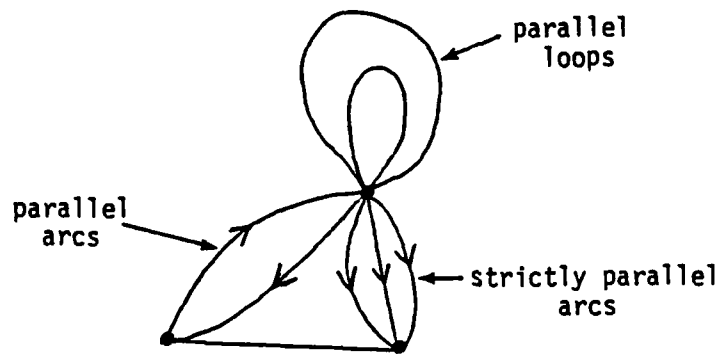
(a) Parallel edges, arcs, and loops.
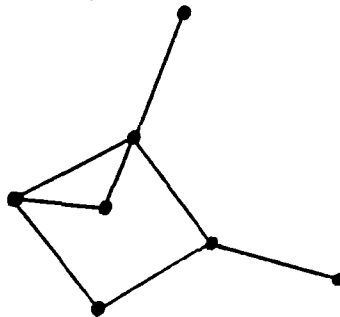
(b) A simple graph.

Figure 4. Parallel edges and simple graphs.

$V(G_1)$ = $\{A,B,C,D,E,F\}$

$E(G_1)$ = $\{a,b,c,d,e,f,g,h,i\}$
= $\{(A,B), (B,C), (C,D), (D,F), (C,F),$
$(C,E), (B,E), (A,F), (A,E)\}$
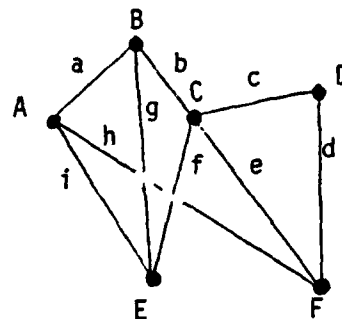
(a) graph G_1 .
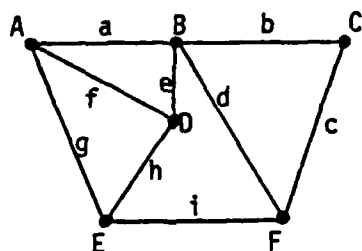
(b) graph G_2 .

$V(G_2)$ = $\{A,B,C,D,E,F\}$

$E(G_2)$ = $\{a,b,c,d,e,f,g,h,i\}$
= $\{(A,B), (B,C), (C,F), (B,F), (B,D),$
$(A,D), (A,E), (D,E), (E,F)\}$

(c) THE ISOMORPHISM:

vertices		edges	
1-1		1-1	
$V(G_1) \longleftrightarrow V(G_2)$		$E(G_1) \longleftrightarrow E(G_2)$	
A	E	a	g
B	A	b	a
C	B	c	b
D	C	d	c
E	D	e	d
F	F	f	e
		g	f
		h	i
		i	h

Figure 5. Isomorphic graphs G_1 and G_2 .

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A *plane graph* is a geometric graph in a plane. A *planar graph* is a graph that is isomorphic to a plane graph. A plane graph divides the plane into *regions*, one of which is infinite in extent. Using stereographic projection or inversion, it is possible to redraw a plane graph so that any desired region is the unbounded one (Ref. 2).

All of the figures depicting graphs in this report are actually geometric graphs which are geometric realizations of graphs having the properties illustrated. Although they are all drawn in a plane, only those figures in which all edges meet only at vertices are plane graphs.

An *edge progression* is a finite sequence of (not necessarily distinct) edges such that one end point of the first edge is also an end point of the second, the remaining end point of the second is also an end point of the third, etc. (Fig. 6). The edge progression is *closed* if the remaining end point of the first edge is the same vertex as the remaining end point of the last, and *open* otherwise. A *chain (circuit) progression* is an open (closed) edge progression having no repeated edges (Fig. 7), and a *chain (circuit)* is a set of edges which, if properly ordered, form a chain (circuit) progression. A *tree* is a graph which contains no circuits. In a geometric graph, a chain (circuit) is a set of edges which form a open (closed) curve. The terms *arc*, *path*, and *cycle* replace the terms edge, chain, and circuit, respectively, when the graph is a digraph (Fig. 8), but frequently the terms are used interchangeably, with their precise meaning indicated by the graph under consideration.

The *degree* of a vertex is the number of edges with which the vertex is incident, with loops counted twice. A *contraction* of a graph is the removal of a vertex V of degree two, replacing its two incident edges (V_1, V) and (V, V_2) by one edge (V_1, V_2) (Fig. 9). Two graphs are *conformal*, or *isomorphic to within vertices of degree two*, if they are isomorphic or can be transformed into isomorphic graphs by contractions (Fig. 10). An *elementary contraction* is the deletion of a vertex V and an edge (V, W), replacing all other edges (U, V) incident with V by edges (U, W) (Fig. 11).
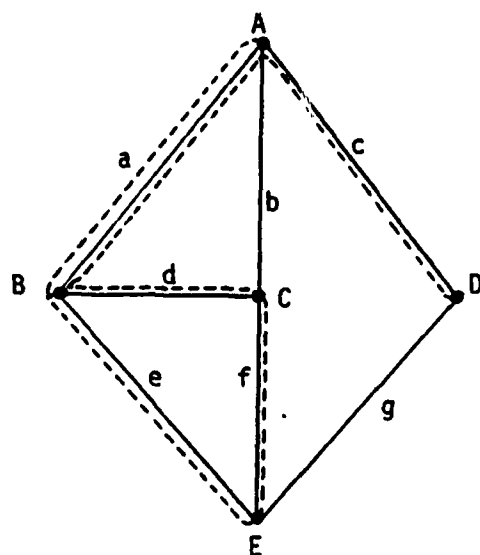
edge progression: (A,B), (B,E), (E,C), (C,B), (B,A), (A,D)

= a,e,f,d,a,c.

Figure 6. Edge progression in a graph.
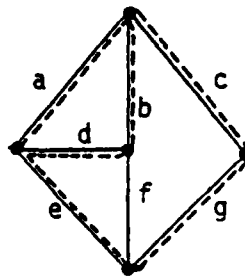
(a) Chain progression a,b,d,e,g,c.
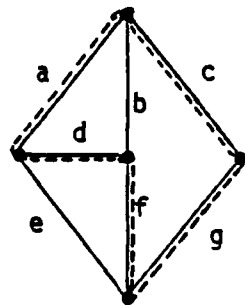
(b) Circuit progression a,d,f,q,c.

Figure 7. Chain and circuit progressions in a graph.

(a) arc progression:

e,g,c,a,e,f
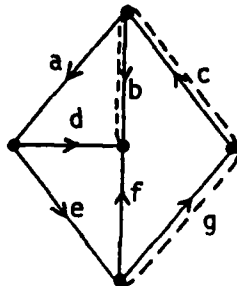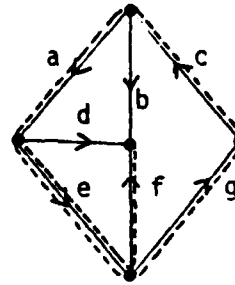
(b) path progression:

g,c,b

(c) cycle progression:

c,a,e,g
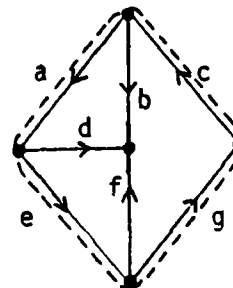
Figure 8. Progressions in a digraph.
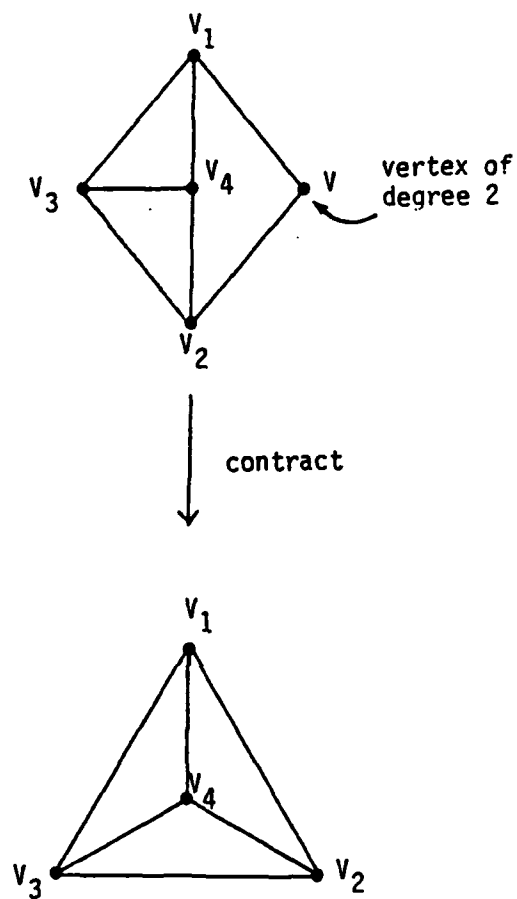
Figure 9. Contraction of a graph.

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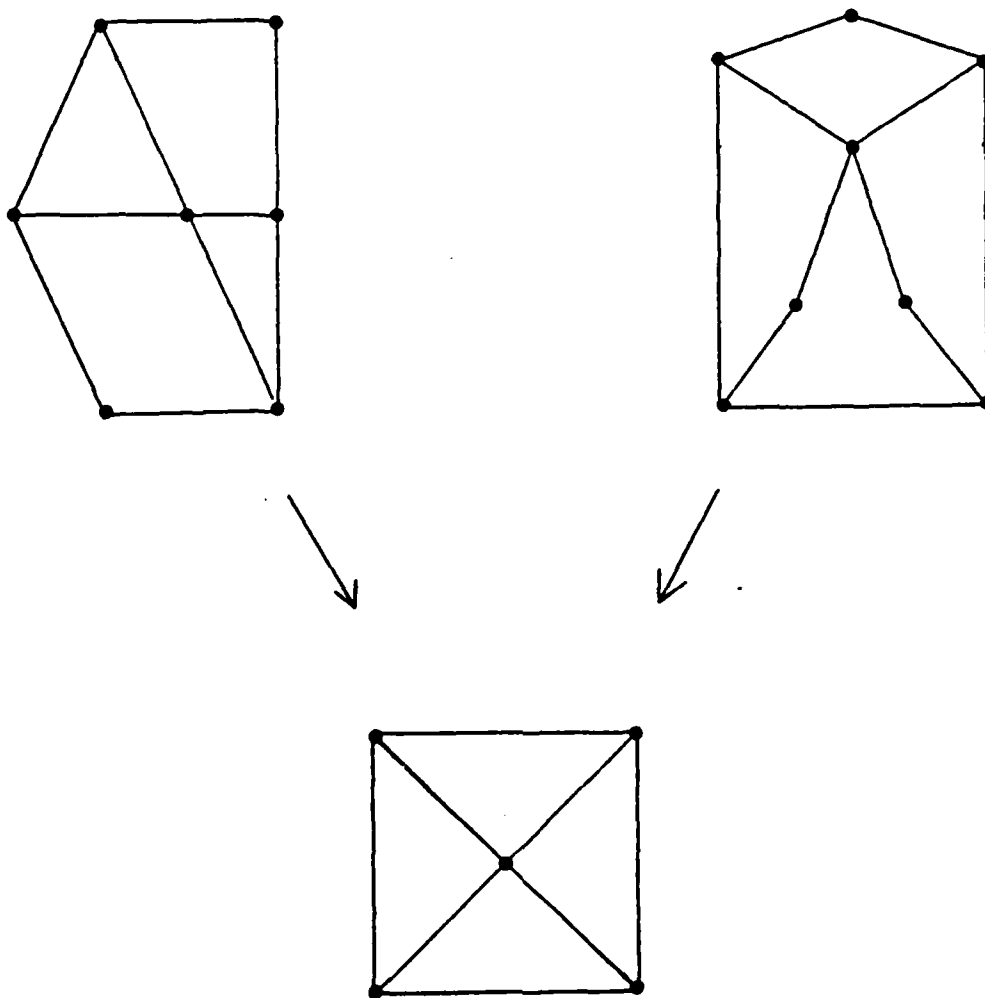
Figure 10. Conformal graphs.

15
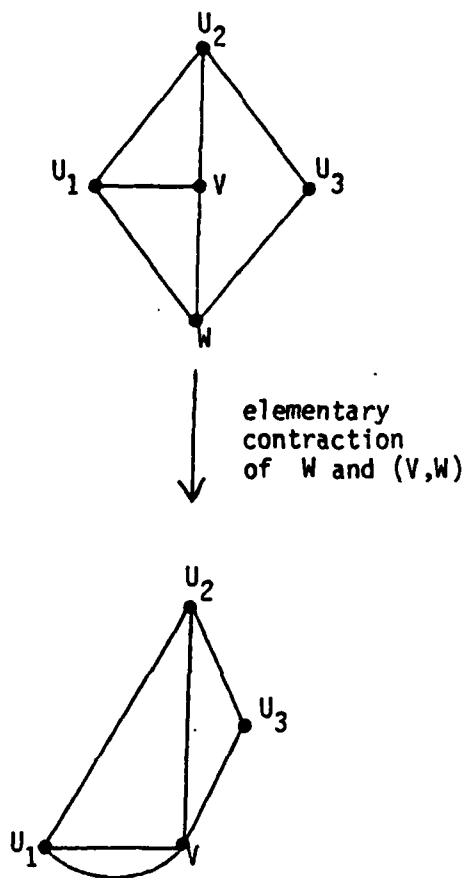
Figure 11. Elementary contraction of a graph.

Associated with a graph comprising n vertices and m edges are several matrices. The *adjacency matrix* A(G) = (A$_{ij}$) is an n x n matrix defined by

$$A_{ij} = \begin{cases} 1 & \text{if vertices } i \text{ and } j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

The adjacency matrix differs from the node-node matrix of electrical circuit theory only on the diagonal, where 0's replace nonzero entries representing self-connection (Ref. 4).

The *degree matrix* D is a diagonal matrix with

D$_{ii}$ = degree of vertex i, for i = 1, ..., n.

There is also ar n x m *incidence matrix* I(G) = (I$_{ij}$), whose entries are given by

$$I_{ij} = \begin{cases} -1 & \text{if vertex } i \text{ is the initial end point of edge } j, \\ 1 & \text{if vertex } i \text{ is the terminal end point of edge } j, \\ 0 & \text{otherwise.} \end{cases}$$

Unlike the node-branch matrix, the incidence matrix distinguishes between initial and terminal vertices (Ref. 4).

These matrices are related by the matrix equation

$$I \cdot I^T = D - A$$

In Figure 12a the graph in Figure 6 has been relabelled to construct the matrices of adjacency, degree and incidence (Figs. 12b, c and d).

$$A(G) \;=\; \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$
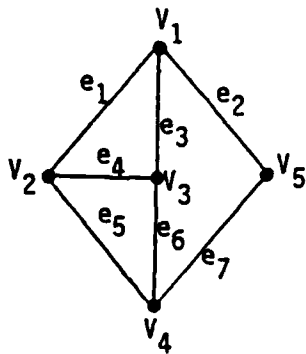
(a) Figure 7 after relabelling.

(b) Adjacency matrix.

$$D(G) \;=\; \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$I(G) \;=\; \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

(c) Degree matrix.

(d) Incidence matrix.

Figure 12. Matrices of a graph.

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Note that every column of $I(G)$ contains at most two nonzero elements since every edge has two end points (not necessarily distinct). A matrix is said to have *graphic form* if every column contains at most two nonzero elements.

Two topological invariants of a graph are used in this report. A *component* of a graph is the largest subset of the vertex set with the property that no vertex of that subset is joined by an edge to any vertex not in that subset. Topologically, a component is a maximal connected subset of the graph. For any graph G comprising n vertices, m edges, and p components,

$$c(G) = m - n + p$$

and

$$c*(G) = n - p$$

$c(G)$ is called the *cycle rank, circuit rank,* or *cyclomatic number.*
$c*(G)$ is called the *cutset rank,* or *cocycle rank.*

III. SOME CRITERIA FOR PLANARITY

Two classes of graphs which have been studied extensively have been named K_n and $K_{r_1,\ldots,r_m}$. K_n is the *complete graph on n vertices*. It is the largest possible simple graph with n vertices, as its edge set consists of all possible pairs of vertices. The symbol K_5 is the smallest nonplanar complete graph (Fig. 13).

The $K_{r_1,\ldots,r_m}$ is called the *complete m-partite graph*. Its vertex set is partitioned into m disjoint subsets, with the ith subset containing r_i vertices. Every vertex in a particular subset is connected by an edge to every vertex not in that subset, but no two vertices in the same subset are so joined. Figure 14 shows $K_{3,3}$, the smallest nonplanar complete bipartite graph. Note the two subsets of the six vertices. Each subset contains three vertices. Each vertex has degree three, because it is joined to every vertex in the complementary subset.

The existence of these two nonplanar graphs yields one of the most useful criteria for distinguishing between planar and nonplanar graphs by inspection:

CRITERION 1:   A necessary and sufficient condition for
              a graph to be planar is that it contains
              no subgraph conformal to K_5 or $K_{3,3}$.

Figure 15 presents a sample EMT. The dual graph, an ISD, is shown in Figure 16. Note that the number of crossings *appears* to be minimized, but it is not clear whether or not the graph is planar. Application of this criterion establishes that the ISD is nonplanar because of the heavy lines tracing out $K_{3,3}$.

Although it contains subgraphs conformal to K_5 and $K_{3,3}$, the Peterson graph, shown in Figure 17, can also be reduced to K_5 by a sequence of elementary contractions. This illustrates another criterion for planarity:
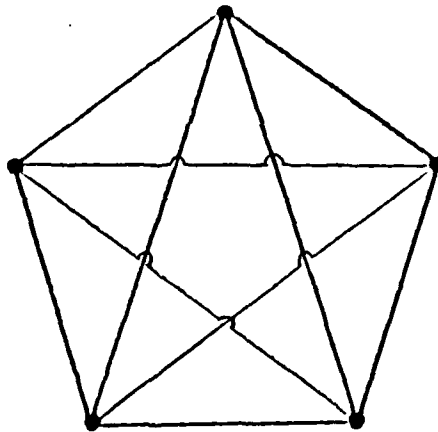
Figure 13. K_5, the smallest nonplanar complete graph.
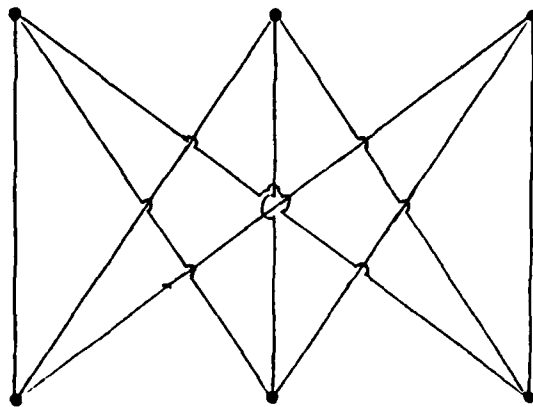
Figure 14. $K_{3,3}$, the smallest nonplanar complete bipartite graph.
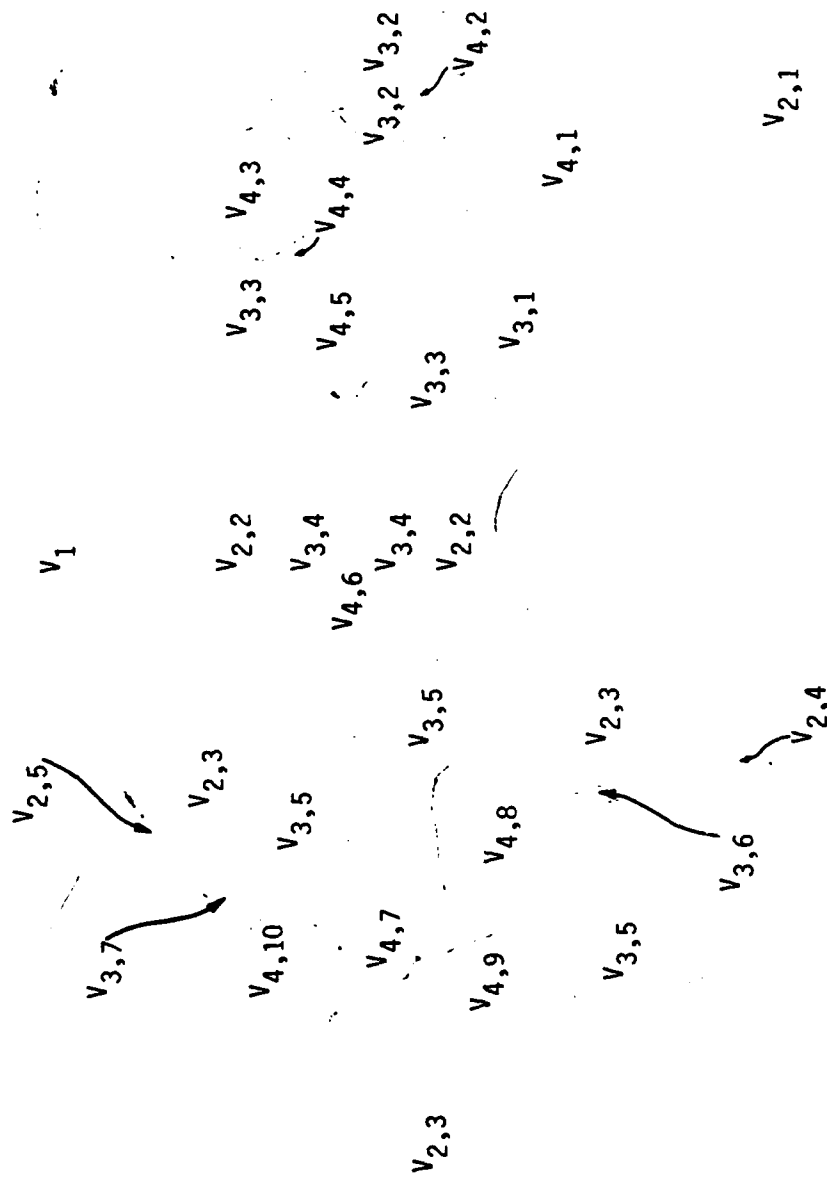
$V_{\lambda,\tau} = \tau^{\underline{th}}$ elementary volume of $\lambda^{\underline{th}}$ sublayer ($= \lambda^{\underline{th}}$ layer)

Figure 15. A sample electromagnetic topology.
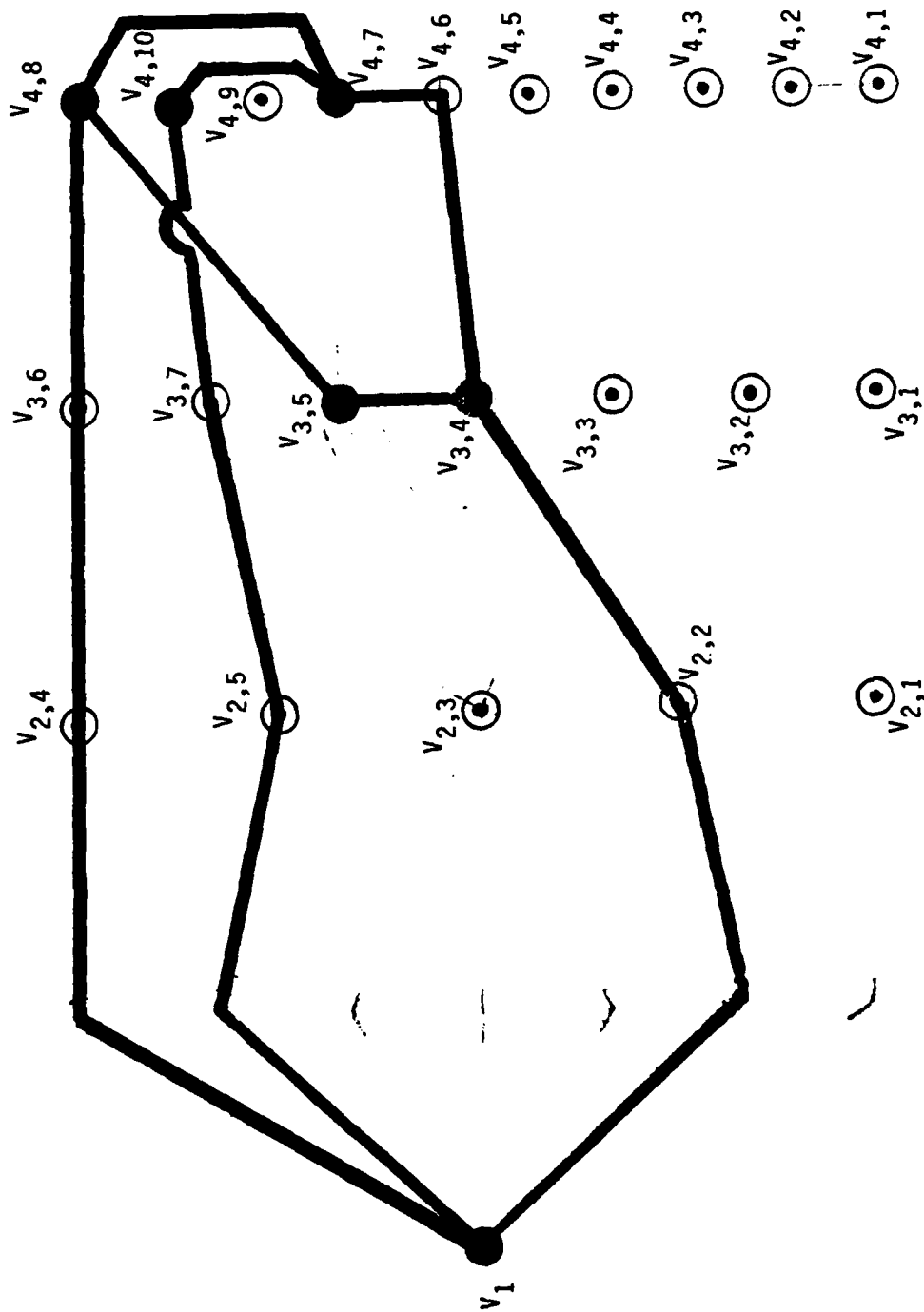
Figure 16. The dual graph (ISD) of Figure 15.

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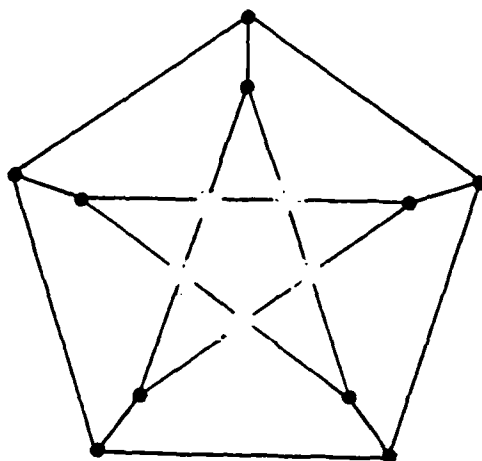
(a) The Peterson graph.

five
elementary
contractions
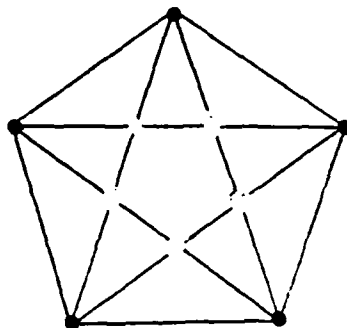
(b) K_5

Figure 17. The Peterson graph.

CRITERION 2:   A graph is planar if and only if it
               contains no subgraphs contractable to
               K_5 or $K_{3,3}$ by means of a sequence
               of elementary contractions.

A *dual graph* of a graph G may be defined in several ways. A planar
graph possesses a *geometric dual* G* such that for each region of G,
including the infinite region, there is a vertex of G*. An edge is drawn
between two vertices of G* if the corresponding regions of G are con-
tiguous (have a common edge as part of their boundary). In this way edges
of G are placed in one-to-one correspondence with edges of G*. The graph
in Figure 6 is an example of a plane graph, shown in Figure 18a with its
geometric dual. The geometric dual of the geometric dual of a plane graph is
isomorphic to the plane graph (Fig. 18b). Note the correspondence between
vertices of degree two and parallel edges.

A *cut-set* of a graph is a disconnecting set (set of edges whose removal
disconnects the graph) consisting of all the edges that join a specified set
of vertices with the complementary set of vertices. A cut-set containing no
proper subsets which are also cut-sets is called a minimal cut-set, proper
cut-set, or cocycle. The graph in Figure 6 has been dissected in Figure 19
to show its 15 cocycles.

A graph G has an *abstract dual* G* if there is a one-to-one corres-
pondence between edges of G and those of G* with the property that a set
of edges of G forms a circuit in G if and only if the corresponding set
of edges in G* forms a cut-set in G*. It has been shown that if G is an
abstract dual of G* then G* is an abstract dual of G. Furthermore, if
G is a planar graph with geometric dual G* then G* is an abstract dual
of G.

CRITERION 3:   A graph is planar if and only if it has
               an abstract dual.
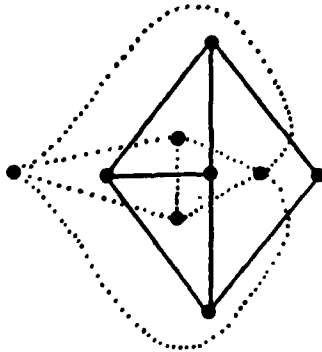
(a) A plane graph (———) and its geometric dual (·····).
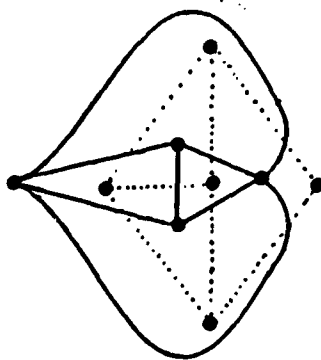
(b) The geometric dual (———) and its geometric dual (·····).

Figure 18. The geometric dual of a plane graph.
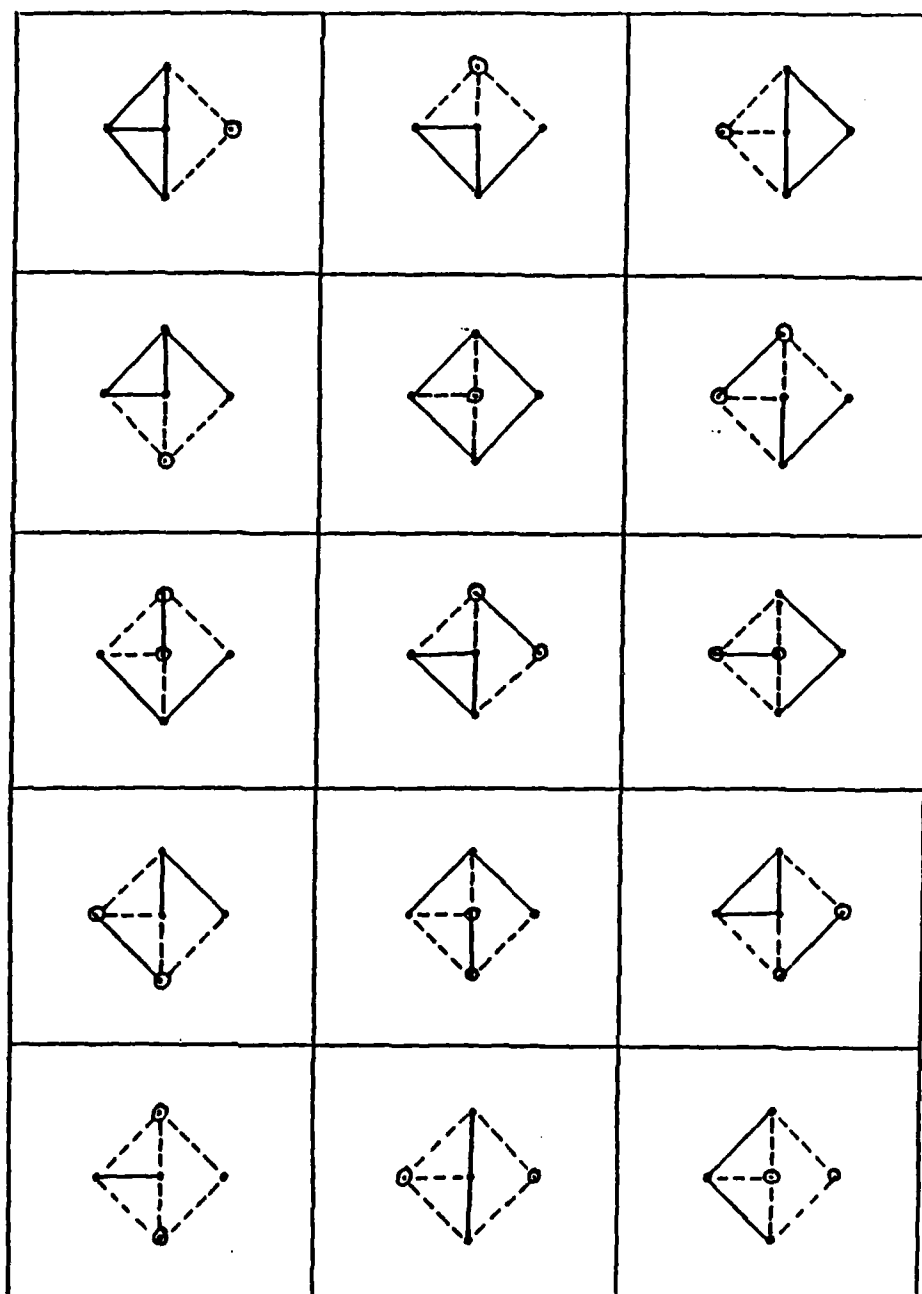
Figure 19. Cocycles of the graph in Figure 12a.

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IV. AN ALGORITHM TO DETERMINE PLANARITY

The listed criteria are useful for visually inspecting small graphs or large symmetrical graphs but, in general, an algorithmic determination of planarity is desired. By working with the vector space associated with the incidence matrix, one can develop an algebraic criterion which depends on the proper choice of basis.

Let G be a graph with n vertices, m edges, and p components. Without loss of generality G may be assumed to be a simple connected graph such that every edge lies on at least one cycle, for the planarity of a graph is not affected by the addition or deletion of multiple edges, loops, or tree subgraphs, and a disconnected graph is planar if, and only if, each of its components is planar. (For an EMT drawn to elementary volume level, however, this assumption is misleading: it may be true that even though the decomposition of each sublayer into elementary volumes results in a dual planar subgraph, the union of all such subgraphs requires nonplanar connecting edges. The problem arises because sublayers are pairwise disjoint, but elementary volumes are not, and the ISD typically consists of only one component.)

The incidence matrix assigns an initial and terminal vertex to each edge. To associate a cycle c with an m-vector C, an orientation is assigned to the cycle. As the cycle is traversed according to this orientation, the directions of the edges may or may not agree with the direction of travel, and are said to contribute positively or negatively, accordingly. Then the m-vector $C = (C_1, \ldots, C_m)$ is defined:

$$C_i = \begin{cases} 1 & \text{if edge } i \text{ contributes positively to the cycle,} \\ -1 & \text{if edge } i \text{ contributes negatively to the cycle,} \\ 0 & \text{otherwise.} \end{cases}$$

Figure 20a shows the graph of Figure 12a after contraction of vertex V_4. This is K_4, the complete graph on four vertices. Two cycles and their associated 6-vector are given in Figure 20b. Note that the product of each 6-vector with the incidence matrix (Fig. 20c) is zero (Fig. 20d).
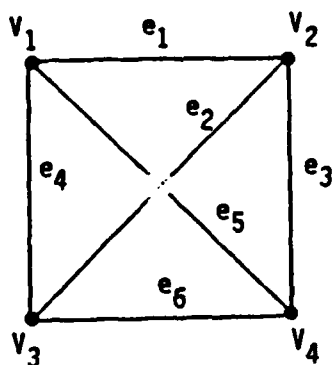
cycle   6-vector

$\{e_1, e_2, e_6, e_5\}$ $C_1 = (-1, -1, 0, 0, 1, -1)$.

$\{e_2, e_6, e_3\}$ $C_2 = (0, -1, 1, 0, 0, -1)$

(a) K_4, the complete graph on four vertices.

(b)   Two cycles in K_4 and their associated 6-vectors.

(c)   incidence matrix $I(K_4) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$

(d) $\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} I(K_4)^T = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} = 0$

Figure 20. Correspondence between m-vectors and cycles.

Not every m-vector C corresponds to a cycle of G, but it can be shown that C corresponds to some cycle, or set of cycles, if and only if $C \cdot I(G)^T = 0$. If C_0 is the set of all m-vectors representing one or more cycles, then C_0 is the null space of $I(G)$, and the dimension of C_0 is $c = c(G) = m - n + 1$, the cycle rank defined above with $p = 1$.

The criterion for planarity of G may now be expressed in terms of the *cycle matrix* whose c rows correspond to the elements of a basis for C_0, and whose m columns correspond to the edges of G. The graph formed from the cycle matrix by construction of the basis cycles is identical to G, since every edge is a cycle edge, and the basis generates all cycles.

CRITERION 4: A graph is planar if and only if there is
a cycle matrix for it having graphic form.

A basis for C_0 consists of c m-vectors. Given any basis, all cycles of G may be obtained by taking all possible combinations of the original c m-vectors. Thus, there are

$$2^c - 1 = \binom{c}{1} + \binom{c}{2} + \cdots + \binom{c}{c}$$

cycles from which a basis may be chosen, so at most $\binom{2^c - 1}{c}$ different cycle matrices must be considered. The following table summarizes the relationship between c, $2^c - 1$, and $\binom{2^c - 1}{c}$ for small values of c.

cycle rank $(c = m-n+1)$	# of cycles $(2^c - 1)$	# of matrices $\binom{2^c - 1}{c}$
2	3	3
3	7	35
4	15	1365
5	31	169,911
6	63	6.8×10^7
7	127	9.0×10^{10}
8	255	4.0×10^{14}
9	511	6.1×10^{18}

Once a basis for C_0 has been found, all other bases may be derived. The first basis may be determined by the use of a *spanning tree*, a tree sub-graph of G having n vertices and n-1 edges. For each edge e of G which is not an edge of T, there is a unique cycle in G containing e and edges of T. Figure 21 shows a spanning tree of the graph in Figure 12a (Fig. 21a) and the unique cycles corresponding to the edges not in the tree (Fig. 21b). By taking all three such edges, a basis for C_0 is obtained (Fig. 21c).

The search for a spanning tree proceeds by deleting edges from G. Since every spanning tree of G contains n-1 edges, there are $\binom{m}{n-1}$ possibilities. The number of spanning trees of G is a topological invariant $\kappa(G)$ called the *complexity* of G. Two related matrix formulas can be computed to obtain $\kappa(G)$:

$$(1) \quad \kappa(G) \ J \ = \ \text{Adj} \ (D - A),$$

and

$$(2) \quad \kappa(G) \ = \ \frac{1}{n^2} \ \det \ (J + D - A),$$

$$\text{where} \quad J \ = \ \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

The first formula states that every cofactor of $D - A$ is equal to $\kappa(G)$. The second formula is a consequence of the first.

For the graph in Figure 12a,

$$J + D - A \ = \ \begin{pmatrix} 4 & 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 & 1 \\ 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 1 & 0 & 3 \end{pmatrix} \quad , \text{ so } \kappa(G) \ = \ 24.$$

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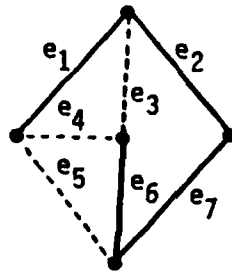
(a) A spanning tree of the graph in Figure 12a.
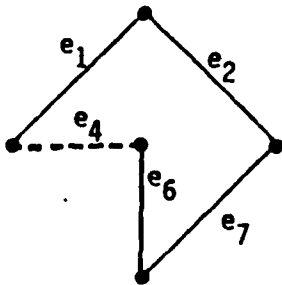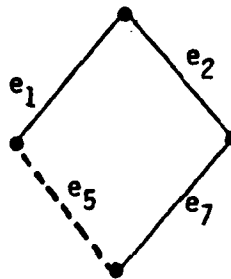
$c_1 = (1,-1,0,1,0,1,1)$ $c_2 = (1,-1,0,0,1,0,1)$ $c_3 = (0,-1,1,0,0,1,1)$
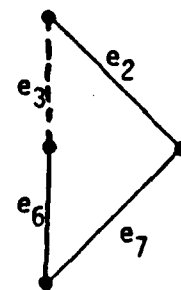
(b) Unique circuits induced by non-tree edges.

(c) cycle matrix = $\begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$

Figure 21. Spanning tree and cycle matrix of a graph.

There is an alternative method of obtaining spanning trees and cycles which may be written down in terms of matrix operations. Any square submatrix of the incidence matrix $I(G)$ of a graph G has determinant equal to -1, 0, or +1. This fact leads to the following theorem (Ref. 5):

"Let U be a subset of $E(G)$ containing $n-1$ edges. Let $I(U)$ denote an $(n-1) \times (n-1)$ submatrix of $I(G)$, consisting of the intersection of those $n-1$ columns of $I(G)$ corresponding to the edges in U and any set of $n-1$ rows of $I(G)$. Then $I(U)$ is non-singular if any only if the subgraph of G having edge set U is a spanning tree of G."

The graph in Figure 12a is shown in Figure 22a, together with its incidence matrix in Figure 22b. The five 4×4 submatrices obtained from the first four columns of $I(G)$ are all singular, since the chain (e_1, e_2, e_3, e_4) contains a cycle (Fig. 22c). However, if the first column is replaced by the fifth column, then all five of the corresponding submatrices are nonsingular (Fig. 22d), showing that (e_2, e_3, e_4, e_5) is a spanning tree.

For this graph, there are seven cycles from which three may be chosen to form a cycle basis. The cycles, labelled $C_1, C_2, \ldots, C_7$, have been drawn in Figure 23, and the 35 possibilities for bases have been written out in Figure 24. The four dependent cycles which are induced (by addition) by each basis are listed below the three cycles forming the basis. Among the seven triplets which do not span the space, each edge occurs three times, suggesting that there is no preferred labelling of the cycles which reduces the number of combinations which must be considered. Application of criterion 4 to the cycle matrices separates the candidates into three classes, denoted in Figure 24 by:

(P) The cycle matrix has graphic form, proving planarity,

(N) Some column of the matrix contains three nonzero elements (no conclusion),

and (O) Some column of the matrix contains three zeros (not a basis).
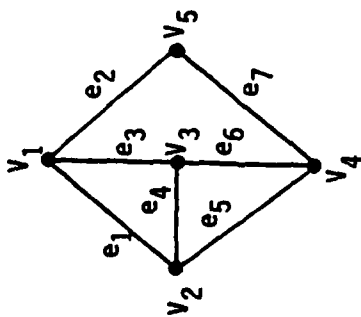
(a) graph in Figure 12a.

$$I(G) = \begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 \end{vmatrix}$$

(b) incidence matrix.

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} = 0$$

(c) submatrices using columns 1,2,3,&4.

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} = 0 \qquad \begin{vmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} = 1 \qquad \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 \end{vmatrix} = -1$$

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{vmatrix} = 1 \qquad \begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{vmatrix} = -1$$

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} = -1$$

(d) submatrices using columns 2,3,4,&5.

Figure 22. Finding spanning trees using Biggs' Theorem (Ref. 5).

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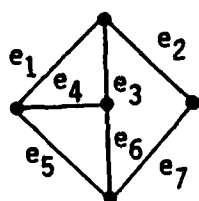
$$c_1 = (1,-1,0,0,1,0,1)$$
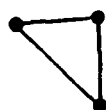
$$c_2 = (0,-1,1,0,0,1,1) \qquad c_3 = (0,0,0,1,-1,1,0)$$

$$c_4 = (1,0,-1,1,0,0,0)$$
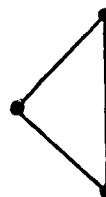
$$c_5 = (1,0,-1,0,1,-1,0)$$
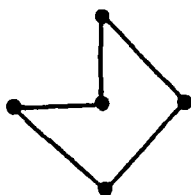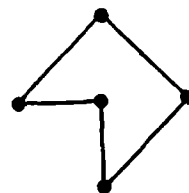
$$c_6 = (0,-1,1,-1,1,0,1) \qquad c_7 = (1,-1,0,1,0,1,1)$$

Figure 23. Cycles of the graph in Figure 12a.

$C_1 \cdot C_2 \cdot C_3$ (P)

$C_1 + C_2 = C_5$
$C_1 + C_3 = C_7$
$C_2 + C_3 = C_6$
$C_1 + C_2 + C_3 = C_4$

$C_1 \cdot C_2 \cdot C_4$ (P)

$C_1 + C_2 = C_5$
$C_1 + C_4 = C_6$
$C_2 + C_4 = C_7$
$C_1 + C_2 + C_4 = C_3$

$C_1 \cdot C_2 \cdot C_5$ (O)

$C_1 + C_2 = C_5$
$C_1 + C_5 = C_2$
$C_2 + C_5 = C_1$
$C_1 + C_2 + C_5 = C_0$

$C_1 \cdot C_2 \cdot C_6$ (N)

$C_1 + C_2 = C_5$
$C_1 + C_6 = C_4$
$C_2 + C_6 = C_3$
$C_1 + C_2 + C_6 = C_7$

$C_1 \cdot C_2 \cdot C_7$ (N)

$C_1 + C_2 = C_5$
$C_1 + C_7 = C_3$
$C_2 + C_7 = C_4$
$C_1 + C_2 + C_7 = C_6$

$C_1 \cdot C_3 \cdot C_4$ (P)

$C_1 + C_3 = C_7$
$C_1 + C_4 = C_6$
$C_3 + C_4 = C_5$
$C_1 + C_3 + C_4 = C_2$

$C_1 \cdot C_3 \cdot C_5$ (N)

$C_1 + C_3 = C_7$
$C_1 + C_5 = C_2$
$C_3 + C_5 = C_4$
$C_1 + C_3 + C_5 = C_6$

$C_1 \cdot C_3 \cdot C_6$ (N)

$C_1 + C_3 = C_7$
$C_1 + C_6 = C_4$
$C_3 + C_6 = C_2$
$C_1 + C_3 + C_6 = C_5$

$C_1 \cdot C_3 \cdot C_7$ (O)

$C_1 + C_3 = C_7$
$C_1 + C_7 = C_3$
$C_3 + C_7 = C_1$
$C_1 + C_3 + C_7 = C_0$

$C_1 \cdot C_4 \cdot C_5$ (N)

$C_1 + C_4 = C_6$
$C_1 + C_5 = C_2$
$C_4 + C_5 = C_3$
$C_1 + C_4 + C_5 = C_7$

$C_1 \cdot C_4 \cdot C_6$ (O)

$C_1 + C_4 = C_6$
$C_1 + C_6 = C_4$
$C_4 + C_6 = C_1$
$C_1 + C_4 + C_6 = C_0$

$C_1 \cdot C_4 \cdot C_7$ (N)

$C_1 + C_4 = C_6$
$C_1 + C_7 = C_3$
$C_4 + C_7 = C_2$
$C_1 + C_4 + C_7 = C_5$

$C_1 \cdot C_5 \cdot C_6$ (N)

$C_1 + C_5 = C_2$
$C_1 + C_6 = C_4$
$C_5 + C_6 = C_7$
$C_1 + C_5 + C_6 = C_3$

$C_1 \cdot C_5 \cdot C_7$ (N)

$C_1 + C_5 = C_2$
$C_1 + C_7 = C_3$
$C_5 + C_7 = C_6$
$C_1 + C_5 + C_7 = C_4$

$C_1 \cdot C_6 \cdot C_7$ (N)

$C_1 + C_6 = C_4$
$C_1 + C_7 = C_3$
$C_6 + C_7 = C_5$
$C_1 + C_6 + C_7 = C_2$

$C_2 \cdot C_3 \cdot C_4$ (P)

$C_2 + C_3 = C_6$
$C_2 + C_4 = C_7$
$C_3 + C_4 = C_5$
$C_2 + C_3 + C_4 = C_1$

$C_2 \cdot C_3 \cdot C_5$ (N)

$C_2 + C_3 = C_6$
$C_2 + C_5 = C_1$
$C_3 + C_5 = C_4$
$C_2 + C_3 + C_5 = C_7$

$C_2 \cdot C_3 \cdot C_6$ (O)

$C_2 + C_3 = C_6$
$C_2 + C_6 = C_3$
$C_3 + C_6 = C_2$
$C_2 + C_3 + C_6 = C_0$

$C_2 \cdot C_3 \cdot C_7$ (N)

$C_2 + C_3 = C_6$
$C_2 + C_7 = C_4$
$C_3 + C_7 = C_1$
$C_2 + C_3 + C_7 = C_5$

$C_2 \cdot C_4 \cdot C_5$ (N)

$C_2 + C_4 = C_7$
$C_2 + C_5 = C_1$
$C_4 + C_5 = C_3$
$C_2 + C_4 + C_5 = C_6$

$C_2 \cdot C_4 \cdot C_6$ (N)

$C_2 + C_4 = C_7$
$C_2 + C_6 = C_3$
$C_4 + C_6 = C_1$
$C_2 + C_4 + C_6 = C_5$

$C_2 \cdot C_4 \cdot C_7$ (O)

$C_2 + C_4 = C_7$
$C_2 + C_7 = C_4$
$C_4 + C_7 = C_2$
$C_2 + C_4 + C_7 = C_0$

$C_2 \cdot C_5 \cdot C_6$ (N)

$C_2 + C_5 = C_1$
$C_2 + C_6 = C_3$
$C_5 + C_6 = C_7$
$C_2 + C_5 + C_6 = C_4$

$C_2 \cdot C_5 \cdot C_7$ (N)

$C_2 + C_5 = C_1$
$C_2 + C_7 = C_4$
$C_5 + C_7 = C_6$
$C_2 + C_5 + C_7 = C_3$

$C_2 \cdot C_6 \cdot C_7$ (N)

$C_2 + C_6 = C_3$
$C_2 + C_7 = C_4$
$C_6 + C_7 = C_5$
$C_2 + C_6 + C_7 = C_1$

$C_3 \cdot C_4 \cdot C_5$ (O)

$C_3 + C_4 = C_5$
$C_3 + C_5 = C_4$
$C_4 + C_5 = C_3$
$C_3 + C_4 + C_5 = C_0$

$C_3 \cdot C_4 \cdot C_6$ (N)

$C_3 + C_4 = C_5$
$C_3 + C_6 = C_2$
$C_4 + C_6 = C_1$
$C_3 + C_4 + C_6 = C_7$

$C_3 \cdot C_4 \cdot C_7$ (N)

$C_3 + C_4 = C_5$
$C_3 + C_7 = C_1$
$C_4 + C_7 = C_2$
$C_3 + C_4 + C_7 = C_6$

$C_3 \cdot C_5 \cdot C_6$ (N)

$C_3 + C_5 = C_4$
$C_3 + C_6 = C_2$
$C_5 + C_6 = C_7$
$C_3 + C_5 + C_6 = C_1$

$C_3 \cdot C_5 \cdot C_7$ (N)

$C_3 + C_5 = C_4$
$C_3 + C_7 = C_1$
$C_5 + C_7 = C_6$
$C_3 + C_5 + C_7 = C_2$

$C_3 \cdot C_6 \cdot C_7$ (N)

$C_3 + C_6 = C_2$
$C_3 + C_7 = C_1$
$C_6 + C_7 = C_5$
$C_3 + C_6 + C_7 = C_4$

$C_4 \cdot C_5 \cdot C_6$ (N)

$C_4 + C_5 = C_3$
$C_4 + C_6 = C_1$
$C_5 + C_6 = C_7$
$C_4 + C_5 + C_6 = C_2$

$C_4 \cdot C_5 \cdot C_7$ (N)

$C_4 + C_5 = C_3$
$C_4 + C_7 = C_2$
$C_5 + C_7 = C_6$
$C_4 + C_5 + C_7 = C_1$

$C_4 \cdot C_6 \cdot C_7$ (N)

$C_4 + C_6 = C_1$
$C_4 + C_7 = C_2$
$C_6 + C_7 = C_5$
$C_4 + C_6 + C_7 = C_3$

$C_5 \cdot C_6 \cdot C_7$ (P)

$C_5 + C_6 = C_7$
$C_5 + C_7 = C_6$
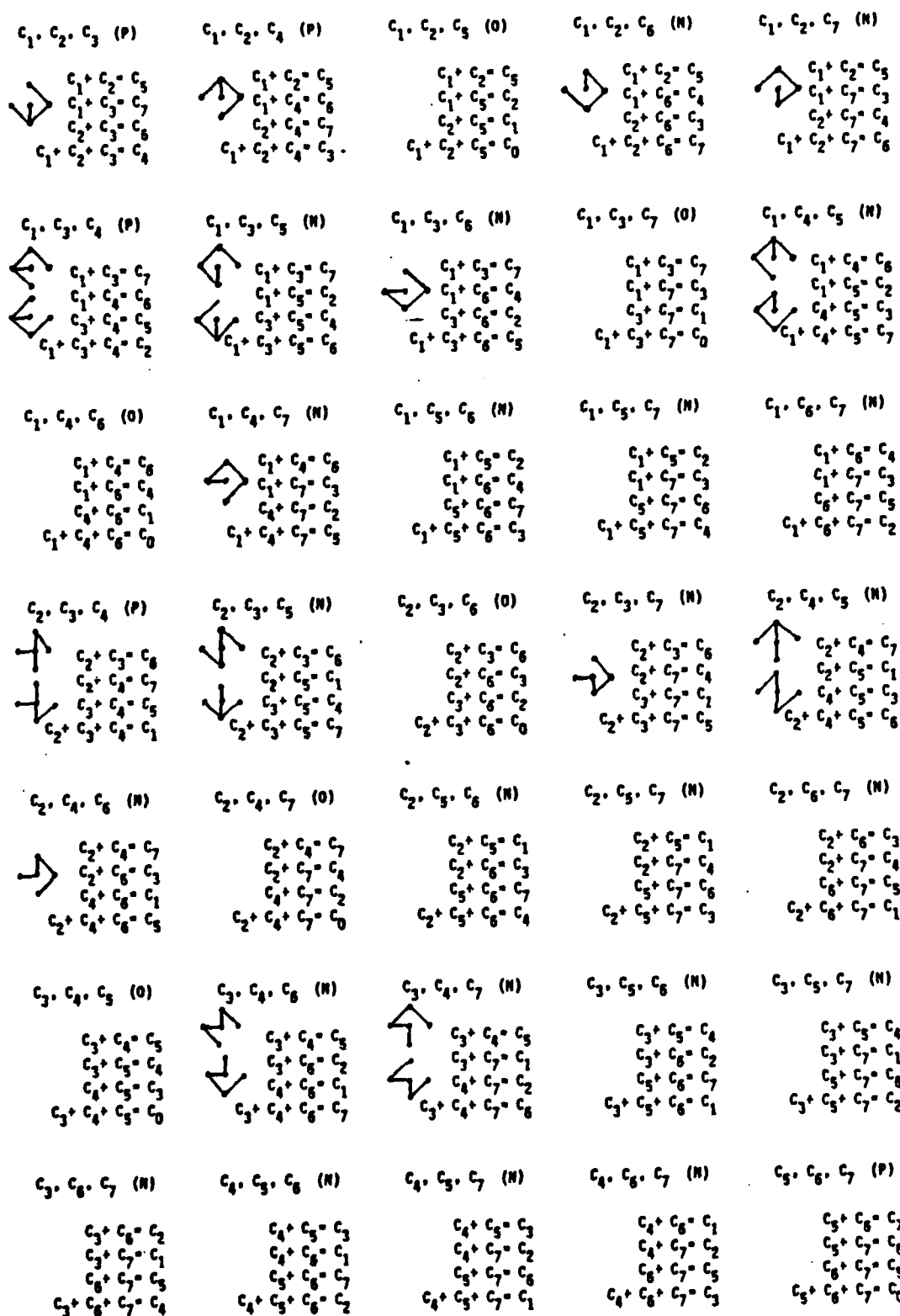$C_6 + C_7 = C_5$
$C_5 + C_6 + C_7 = C_0$

Figure 24.   Cycle bases and spanning trees for Figure 12a.

Also shown in Figure 24 are the spanning trees which induce the cycle bases. In this example, all bases which satisfy criterion 4 for planarity can be induced by a spanning tree. In general, however, neither existence nor uniqueness of a spanning tree generating a given basis is guaranteed. If it could be proved that only bases induced by spanning trees need to be checked, the usefulness of the algorithm would be increased greatly since, in general, the complexity of a graph is a much smaller number than the combinatorial numbers $\binom{2c - 1}{c}$.

V. CONCLUSIONS

In this report, the problem of determining when a graph can be redrawn as a planar graph was addressed. Several criteria were presented and illustrated. One particular criterion, the existence of a cycle matrix having graphic form, was discussed in detail. The computational effort required to solve graphs of practical size is beyond our present and projected computational capabilities. Further theoretical work must be done to improve the efficiency of this algorithm if it is to be useful.

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